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Fuzzy sphere and hyperbolic space from deformation quantization

Isao Kishimoto

Department of Physics, Kyoto University
Kyoto 606-8502, Japan
E-mail: ikishimo@gauge.scphys.kyoto-u.ac.jp

Abstract: We explicitly construct noncommutative $*$ products on circularly symmetric two dimensional space by using the technique of Fedosov's deformation quantization. Especially, on constant curvature spaces i.e. S^2 and H^2, we get su(2) and su(1,1) algebra respectively. These are candidates of $*$ products applicable to noncommutative field theories or noncommutative gauge theories on spaces with nontrivial symplectic structure.

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Contents

1. Introduction

Since the relation between string theory and noncommutative geometry was discussed in [1], noncommutative field theories and noncommutative gauge theories have been investigated enthusiastically from various viewpoints.

Many authors use the Moyal product[1] as noncommutative associative $*$ product for explicit calculations. It corresponds to a constant NS-NS B-field background in flat space in the context of string theory. On the other hand, at least formally, more general $*$ products which may correspond to string theory on nonconstant B-field background in curved space are defined by some authors.[2] However, explicit form of $*$ products other than the Moyal product has been scarcely discussed in physical context.[3]

In this paper, we use the technique of Fedosov's deformation quantization [3] to get explicit forms of $*$ products on nontrivial backgrounds. For simplicity, we investigate $*$ products on circularly symmetric two dimensional spaces. Specifically, we focus on constant curvature spaces S^2, H^2 and $\mathbb{R}^2$, and explicitly construct $*$ products which are different from the Moyal product. We also discuss some physical applications of our $*$ products.

[1]Here we call $* = \exp\left(\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}\right)$ with constant $\theta^{ij} = -\theta^{ji}$ the Moyal product.

[2]Refs. [2, 3], for example.

[3]In [4], nonassociative star product which generalizes [2, 3] is discussed to describe D-brane in curved backgrounds.

2. Construction of $*$ product

Here we review the construction of Fedosov's $*$ product very briefly,[4] and apply this procedure to circularly symmetric two dimensional spaces.

First, for a given symplectic manifold (M, Ω_0), we define the Weyl algebra bundle W which has $\circ$ product of the Moyal type and its abelian connection D with some input parameter. For $\mathrm{Ker}D \subset W$ (which is called flat section W_D), we get a one to one correspondence with $C^\infty(M)[[\hbar]]$, where $\hbar$ is the deformation parameter. We denote the map from $C^\infty(M)[[\hbar]]$ to W_D as Q, and its inverse map as σ. Then Fedosov's $*$ product on $C^\infty(M)[[\hbar]]$ is defined by

$$a_0 * b_0 := \sigma(Q(a_0) \circ Q(b_0)), \qquad a_0, b_0 \in C^\infty(M)[[\hbar]]. \tag{2.1}$$

This is a solution of the problem of deformation quantization, i.e. $*$ is associative and its commutator $[\ ,\ ]_*$ is expanded as

$$[\ ,\ ]_* = i\hbar\{\ ,\ \} + \mathcal{O}(\hbar^2), \tag{2.2}$$

where $\{\ ,\ \}$ is the Poisson bracket with respect to the symplectic form Ω_0.

Now, we apply this procedure to a two dimensional space M with metric

$$ds^2 = e^{\Phi(r)}(dr^2 + r^2 d\theta^2), \tag{2.3}$$

where $\Phi(r)$ is some function of r only (i.e. circularly symmetric space) for simplicity. Its volume form is given by

$$\Omega_0 = e^{\Phi(r)} r dr \wedge d\theta, \tag{2.4}$$

and we identify it with symplectic form. Using Fedosov's procedure with the input[5]

$$\begin{aligned}
\Omega_0 &= \theta^1 \wedge \theta^2 = -\frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j, \\
\theta^1 &= e^{\Phi(r)}dr, \quad \theta^2 = r d\theta, \qquad \omega_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\Omega_1 &= 0, \qquad \nabla = d, \\
\mu &= \frac{1}{3} e^{-\Phi(r)} r^{-1} (y^1)^2 y^2,
\end{aligned} \tag{2.5}$$

we get an abelian connection D as

$$\begin{aligned}
Da &= da + \frac{i}{\hbar}(\mathrm{r} \circ a - a \circ \mathrm{r}), \quad a \in W, \\
\mathrm{r} &= e^{-\Phi(r)} r^{-1} y^1 y^2 \theta^1, \\
\circ &:= \exp\left(-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial y^j}\right), \qquad \omega^{ij} := (\omega^{-1})^{ij}.
\end{aligned} \tag{2.6}$$

[4]See [3, 5] for details.

[5]See [3, 5] for the meaning of ∇, Ω_1, μ. Here we choose these parameters in such a way that the iteration formula ([5, eq.(21)]) which gives an abelian connection is satisfied trivially, i.e. $\nabla \mathrm{r} + (i/\hbar)\mathrm{r} \circ \mathrm{r} = 0$. Then we get $\mathrm{r} = \delta\mu + \delta^{-1}(d(\omega_{ij}y^i\theta^j) - \Omega_1)$ for the input (2.5).

For this abelian connection D, we solve the equation $Da = 0$ and get the map $Q : C^\infty(M)[[\hbar]] \to W_D$ as

$$a = Q(a_0(r,\theta)) = a_0\left(G(r,y^1), \theta + \frac{y^2}{r}\right), \tag{2.7}$$

where $G(r,y^1)$ is given by

$$\int_r^{G(r,y^1)} e^{\Phi(r')} r' dr' = y^1 r\,. \tag{2.8}$$

Then we can define a $*$ product on M by eq. (2.1).

3. S^2 case

In this section we apply the result of section 2 to the case $M = S^2$. We consider 2-sphere S^2 with radius R, which is defined as two dimensional surface embedded in $\mathbb{R}^3$:

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2\,. \tag{3.1}$$

We parametrize the coordinate $X^i, i = 1,2,3$ on S^2 as

$$\begin{aligned} &X^1 = \frac{2R^2 r}{r^2+R^2}\cos\theta\,, \qquad X^2 = \frac{2R^2 r}{r^2+R^2}\sin\theta\,, \qquad X^3 = R\frac{r^2-R^2}{r^2+R^2}\,,\\ &r \geq 0,\ \ 0 \leq \theta \leq 2\pi\,. \end{aligned} \tag{3.2}$$

Then the metric of S^2, $ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2$, is given by

$$ds^2 = \frac{4R^4}{(r^2+R^2)^2}(dr^2 + r^2 d\theta^2)\,, \tag{3.3}$$

and the conformal factor e^Φ of eq. (2.3) is identified as

$$e^{\Phi(r)} = \frac{4R^4}{(r^2+R^2)^2}\,. \tag{3.4}$$

From eqs. (3.4), (2.7) and (2.1), we get the explicit form of our $*$ product on S^2:

$$\begin{aligned} &a_0(r,\theta) * b_0(r,\theta) = \\ &\quad = \left(a_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(r^2+R^2)}{1 - \frac{y^1}{2R^4}r(r^2+R^2)}}, \theta + \frac{y^2}{r}\right) \exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1}\right)\right)\cdot\right.\\ &\qquad \left.\cdot\, b_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(r^2+R^2)}{1 - \frac{y^1}{2R^4}r(r^2+R^2)}}, \theta + \frac{y^2}{r}\right)\right)_{y^1=y^2=0}\,. \end{aligned} \tag{3.5}$$

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By using this definition, we can calculate $*$ product of the S^2 coordinate X^i (3.2). In particular, we have

$$[X^i, X^j]_* = i\frac{\hbar}{R}\varepsilon^{ijk}X^k\,, \tag{3.6}$$

$$X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = R^2\left(1 - \frac{\hbar^2}{4R^4}\right), \tag{3.7}$$

where ε^{ijk} is the antisymmetric tensor with $\varepsilon^{123} = +1$. Eq. (3.6) means that the commutators of X^i's form su(2) algebra which is known as fuzzy sphere algebra, and eq. (3.7) means that its radius is given by $R\sqrt{1-(\hbar^2/4R^4)}$ which is deformed by $\mathcal{O}(\hbar^2)$ from the original radius R of commutative S^2 (3.1). Namely, we have obtained a fuzzy sphere by deforming S^2 using the $*$ product (3.5).

4. H^2 case

In this section we apply the result of §2 to the case $M = H^2$. Calculation is quite similar to the S^2 case (§3). We consider two dimensional hyperbolic space H^2 with radius R, which is defined as two dimensional surface embedded in $\mathbb{R}^{1,2}$:

$$-(Y^0)^2 + (Y^1)^2 + (Y^2)^2 = -R^2\,, \qquad Y^0 > 0\,. \tag{4.1}$$

We parametrize the coordinates $Y^i, i = 0, 1, 2$ on H^2 as

$$\begin{aligned} &Y^0 = R\frac{R^2+r^2}{R^2-r^2}\,, \quad Y^1 = \frac{2R^2 r}{R^2-r^2}\cos\theta\,, \quad Y^2 = \frac{2R^2 r}{R^2-r^2}\sin\theta\,, \\ &0 \le r \le R\,, \quad 0 \le \theta \le 2\pi\,. \end{aligned} \tag{4.2}$$

Then, the metric of H^2, $ds^2 = -(dY^0)^2 + (dY^1)^2 + (dY^2)^2$, and the conformal factor are given respectively by

$$ds^2 = \frac{4R^4}{(R^2-r^2)^2}(dr^2 + r^2 d\theta^2)\,, \tag{4.3}$$

$$e^{\Phi(r)} = \frac{4R^4}{(R^2-r^2)^2}\,. \tag{4.4}$$

From eqs. (4.4), (2.7) and (2.1), we get the explicit form of our $*$ product on H^2:

$$\begin{aligned} &a_0(r,\theta) * b_0(r,\theta) = \\ &\quad = \left(a_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(R^2-r^2)}{1 + \frac{y^1}{2R^4}r(R^2-r^2)}}, \theta + \frac{y^2}{r}\right)\exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1}\right)\right)\cdot\right. \\ &\qquad \left.\cdot\, b_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(R^2-r^2)}{1 + \frac{y^1}{2R^4}r(R^2-r^2)}}, \theta + \frac{y^2}{r}\right)\right)_{y^1=y^2=0}. \end{aligned} \tag{4.5}$$

By using this definition, we obtain the following $*$ products of the H^2 coordinate Y^i (4.2):

$$[Y^0, Y^1]_* = i\frac{\hbar}{R}Y^2\,, \qquad [Y^2, Y^0]_* = i\frac{\hbar}{R}Y^1\,, \qquad [Y^1, Y^2]_* = -i\frac{\hbar}{R}Y^0\,, \tag{4.6}$$

$$-Y^0 * Y^0 + Y^1 * Y^1 + Y^2 * Y^2 = -R^2\left(1 - \frac{\hbar^2}{4R^4}\right). \tag{4.7}$$

Eq. (4.6) means that commutators of Y^i's form su$(1,1)$ algebra which corresponds to isometry of H^2, and eq. (4.7) means that its radius is given by $R\sqrt{1-(\hbar^2/4R^4)}$ which is deformed by $\mathcal{O}(\hbar^2)$ from the original radius R of commutative H^2 (4.1). Namely, we get fuzzy hyperbolic space by deforming H^2 using the $*$ product (4.5).

5. Large R limit and $\mathbb{R}^2$

Here we consider large radius limit of the results of section 3 and section 4. The sectional curvature of S^2 (3.1) (H^2 (4.1)) is $(1/R^2)$ $(-1/R^2)$, which tends to $+0$ (-0) in the limit $R \to \infty$. Therefore they approach the flat space $\mathbb{R}^2$ in the large R limit in the usual commutative picture. How about it from the noncommutative viewpoint?

For comparison, we construct a $*$ product on $\mathbb{R}^2$ following the method of section 2. We adopt as its flat metric

$$ds^2 = 4(dr^2 + r^2 d\theta^2) \tag{5.1}$$

with its front factor 4 chosen so that (5.1) coincides with the large R limit of (3.3) and (4.3). With $e^\Phi = 4$, we get the explicit form of our $*$ product on $\mathbb{R}^2$:

$$\begin{aligned} &a_0(r,\theta) * b_0(r,\theta) = \\ &= \left(a_0\left(\sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r}\right) \exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1}\right)\right)\right. \\ &\qquad \left.\cdot\, b_0\left(\sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r}\right)\right)_{y^1=y^2=0}. \end{aligned} \tag{5.2}$$

Then, we can calculate the $*$ products of the complex coordinate $z := re^{i\theta}$, $\bar{z} := re^{-i\theta}$:

$$\begin{aligned} z * z &= \sqrt{r^4 - \frac{\hbar^2}{16}}e^{2i\theta} = \overline{\bar{z} * \bar{z}}\,, \qquad z * \bar{z} = r^2 - \frac{\hbar}{4}\,, \qquad \bar{z} * z = r^2 + \frac{\hbar}{4}\,, \\ [z, \bar{z}]_* &= -\frac{\hbar}{2}\,. \end{aligned} \tag{5.3}$$

The commutator $[z, \bar{z}]_*$ coincides with that of the usual Moyal product for cartesian coordinates on $\mathbb{R}^2$, but $*$ product itself is different from the Moyal product. This difference comes from ambiguity of deformation quantization.

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We can calculate the commutator $[z, \bar{z}]_*$ also in the S^2 and H^2 cases. For S^2, from eq.(3.5) we get

$$[z, \bar{z}]_* = \frac{-(\hbar/2R^4)(r^2+R^2)^2}{1-\left((\hbar/4R^4)(r^2+R^2)\right)^2} = -\frac{\hbar}{R^4}(R^2+z*\bar{z})(R^2+\bar{z}*z)\,. \tag{5.4}$$

And for H^2, from eq.(4.5) we get

$$[z, \bar{z}]_* = \frac{-(\hbar/2R^4)(R^2-r^2)^2}{1-\left((\hbar/4R^4)(R^2-r^2)\right)^2} = -\frac{\hbar}{2R^4}(R^2-z*\bar{z})(R^2-\bar{z}*z)\,. \tag{5.5}$$

Both eqs.(5.4) and (5.5) are reduced to $[z, \bar{z}]_* = -\frac{\hbar}{2}$ (5.3) as $R \to \infty$. In other words, the $*$ product which we obtained in §2 connects su(2) algebra (or fuzzy S^2) with su(1,1) algebra (or fuzzy H^2) through $R = \infty$.

6. An application

In the previous sections, we explicitly calculated $*$ products by using Fedosov's formulation. They are candidates of $*$ product for defining noncommutative field theory or noncommutative gauge theory on fuzzy S^2, H^2 and $\mathbb{R}^2$.

As an example, we discuss four dimensional noncommutative U(1) gauge theory with one scalar field which is given by the action[6]

$$S = \mathrm{Tr}\left(\frac{1}{4}G^{IJ}G^{KL}F_{IK}*F_{JL} + \frac{1}{2}G^{IJ}D_I\phi * D_J\phi\right). \tag{6.1}$$

We assume that only two dimensional space is noncommutative (1,2 direction), and use a general formulation of noncommutative gauge theory of [5]:

$$\begin{aligned}
G^{IJ} &= \delta^{IJ}\,, \qquad I,J = 1,\cdots,4,\\
F_{IJ} &= \partial_I A_J - \partial_J A_I - i[A_I, A_J]_* - \frac{J_{IJ}}{\hbar}\,, \qquad J_{12} = -J_{21} = 1\,, \quad \text{others} = 0\,,\\
\partial_I &= \frac{i}{\hbar}[-J_{IJ}\tilde{\phi}^J,\ ]_*\,, \qquad I = 1,2\,, \qquad \partial_3 = \frac{\partial}{\partial x^3}\,, \qquad \partial_4 = \frac{\partial}{\partial x^4}\\
D_I\phi &= \partial_I\phi - i[A_I, \phi]_*\,.
\end{aligned} \tag{6.2}$$

Here, $\tilde{\phi}^I$ is the "canonical" noncommutative coordinate satisfying

$$\frac{i}{\hbar}[\tilde{\phi}^1, \tilde{\phi}^2]_* = 1\,. \tag{6.3}$$

Its explicit form is

$$\tilde{\phi}^1 = \frac{2Rr}{\sqrt{r^2+R^2}}\cos\theta\,, \qquad \tilde{\phi}^2 = \frac{2Rr}{\sqrt{r^2+R^2}}\sin\theta \tag{6.4}$$

[6] The symbol Tr is trace for the $*$ product satisfying $\mathrm{Tr} f*g = \mathrm{Tr} g*f$ [3], but we can discuss equations of motion without using the explicit form of the trace.

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for fuzzy S^2 (3.5),

$$\tilde{\phi}^1 = \frac{2Rr}{\sqrt{R^2 - r^2}} \cos\theta \,, \qquad \tilde{\phi}^2 = \frac{2Rr}{\sqrt{R^2 - r^2}} \sin\theta \tag{6.5}$$

for fuzzy H^2 (4.5), and

$$\tilde{\phi}^1 = 2r\cos\theta \,, \qquad \tilde{\phi}^2 = 2r\sin\theta \tag{6.6}$$

for fuzzy $\mathbb{R}^2$ (5.2). The action (6.1) is invariant under noncommutative U(1) gauge transformation:

$$\delta_\lambda A_I = \partial_I \lambda - i[A_I, \lambda]_* \,, \qquad \delta_\lambda \phi = -i[\phi, \lambda]_* \,. \tag{6.7}$$

The equations of motion of (6.1) are

$$D^I F_{IJ} = -i[\phi, D_J\phi]_* \,, \qquad D^I D_I \phi = 0 \,, \tag{6.8}$$

and we obtain a solution by solving the U(1) noncommutative BPS equation:

$$B_I = D_I\phi \,, \quad I = 1,2,3 \,, \qquad \partial_4 = 0, A_4 = 0 \,, \qquad B_I := \frac{1}{2}\varepsilon^{IJK}\left(F_{JK} + \frac{J_{JK}}{\hbar}\right) . \tag{6.9}$$

Under the ansatz

$$\begin{aligned} A_1 + iA_2 &= if_A(l, x^3)(\tilde{\phi}^1 + i\tilde{\phi}^2) \,, \qquad A_3 = 0, \\ \phi &= f(l, x^3) \,, \qquad l := \sqrt{(\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2 + (x^3)^2} \,, \end{aligned} \tag{6.10}$$

eq. (6.9) can be rewritten as

$$\begin{aligned} \partial_3 G^{(m)} - 4\partial_L f^{(m)} &= \sum_{2n+k=m, n\geq 1} \frac{4\partial_L^{2n+1} f^{(k)}}{(2n+1)!} + \sum_{2n+k+k'=m-1} \frac{4G^{(k')}\partial_L^{2n+1} f^{(k)}}{(2n+1)!} \,, \\ \partial_3 f^{(m)} - \partial_L(LG^{(m)}) &= \sum_{2n+k=m, n\geq 1} \frac{\partial_L^{2n+1}(LG^{(k)})}{(2n+1)!} \end{aligned} \tag{6.11}$$

with

$$L := (\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2 \,, \qquad f = \sum_{k=0}^{\infty} \hbar^k f^{(k)} \,, \qquad \left(\frac{1}{\hbar} + f_A\right)^2 = \frac{1}{\hbar^2} + \frac{1}{\hbar}\sum_{k=0}^{\infty} \hbar^k G^{(k)} \,. \tag{6.12}$$

We can solve eq. (6.11) order by order in $\hbar$, and we get

$$\begin{aligned} f &= \frac{g}{l} + \hbar g^2 \left(\frac{2x^3}{l^4} - \frac{1}{l^3}\right) + \hbar^2 \left(\frac{-8g^3x^3}{l^6} - \frac{g}{4l^5} + \left(\frac{5g}{8} + 10g^3\right)\frac{(x^3)^2}{l^7}\right) + \mathcal{O}(\hbar^3) \,, \\ f_A &= \frac{g}{l(l+x^3)} + \hbar g^2 \left(\frac{2}{l^4} - \frac{1}{l^3(l+x^3)} - \frac{1}{2l^2(l+x^3)^2}\right) + +\hbar^2 \Big(\frac{-8g^3}{l^6} + \frac{4g^3}{l^5(l+x^3)} + \\ &\quad + \frac{g^3}{l^4(l+x^3)^2} + \frac{g^3}{2l^3(l+x^3)^3} - \left(\frac{5g}{8} + 10g^3\right)\frac{x^3}{l^7}\Big) + \mathcal{O}(\hbar^3) \,, \end{aligned} \tag{6.13}$$

as a solution such that it becomes the U(1) Dirac monopole in the commutative limit (i.e. $\hbar \to 0$). In the fuzzy $\mathbb{R}^2$ case (6.6), the $\mathcal{O}(\hbar)$ terms coincide with those in [6] which solved the equations of motion with the usual Moyal product.

7. Conclusion and discussion

In this paper we have presented explicit construction of $*$ products on two dimensional constant curvature spaces S^2, H^2 and $\mathbb{R}^2$. We have found that the algebras of the $*$ products represent fuzzy S^2, H^2 and $\mathbb{R}^2$ because the commutators of the $*$ product form su(2), su(1,1) and Heisenberg algebra respectively. The commutators $[z, \bar{z}]_*$ for fuzzy S^2 and H^2 are reduced to that of fuzzy $\mathbb{R}^2$ in the large R limit. In this sense, fuzzy S^2 and H^2 approach to fuzzy $\mathbb{R}^2$ as $R \to \infty$. This is consistent with usual commutative picture.

In section 6 we applied explicit form of our $*$ products to U(1) noncommutative BPS equation (6.9), and obtained its solution to $\mathcal{O}(\hbar^2)$. In eq. (6.9) the $*$ product appears only in the commutator $[\ ,\ ]_*$. Therefore, eq. (6.9) is solved unifiedly for fuzzy S^2, H^2 and $\mathbb{R}^2$ by using "canonical" noncommutative coordinate $\tilde{\phi}^I$ (6.3). In other words, we can get a solution of eq. (6.9) even if the definition of $*$ is different as long as we use "canonical" noncommutative coordinate $\tilde{\phi}^I$ for the $*$ product.

To study the effects of the difference of $*$ products themselves, we should consider noncommutative equations containing "bare" $*$ products. Its typical example is $\phi * \phi = \phi$ which is essentially the equation for noncommutative soliton [7]. Even for the $\mathbb{R}^2$ case, the $*$ product which we get here is different from the usual Moyal product, and hence $\phi \sim \exp(-r^2)$ is *not* a solution[7] of $\phi * \phi = \phi$. It is a future problem to find an explicit solution of it and to investigate its meaning.

For fuzzy S^2, $*$ product is usually defined by using representation matrix of su(2) and spherical harmonic function, and depends on the size of matrix. On the other hand our $*$ product depends on the deformation parameter $\hbar$, so they are very different in appearance. It is also a future problem to study an explicit relation between them. If the relation becomes clear, our $*$ product may give some suggestions to string theory in the literature [8] for example.

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[7] In the case of the Moyal product, this is a solution.

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